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Local Hardy spaces and summability of Fourier transforms

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ABSTRACT

New Wiener amalgam spaces are introduced for local Hardy spaces. A general summability method, the so-called θ-summability is considered for multi-dimensional Fourier transforms. Under some conditions on θ, it is proved that the maximal operator of the θ-means is bounded from the amalgam space $W(h_p^\square, \ell_\infty)$ to $W(L_p, \ell_\infty)$. This implies the almost everywhere convergence of the θ-means for all $f \in W(L_1, \ell_\infty) \supset L_1$.

1. Introduction

Butzer and Nessel [3] and recently Bokor, Schipp, Szili and Vértesi [14,2,18,17], Trigub and Belinsky [21] and Weisz [23–26] considered a general method of summation, the so-called θ-summation, which is generated by a function θ. They proved that if $\hat{\theta}$ can be estimated by a non-increasing integrable function, then the θ-means $\sigma_T^\theta f$ of the Fourier transform of a function $f \in L_p(\mathbb{R})$ $(1 \leqslant p \leqslant \infty)$ converge to f a.e. The multi-dimensional version of this theorem can be found in Stein and Weiss [16].

Feichtinger and Weisz [5] generalized these results and proved that if $\hat{\theta}$ is in a Herz space than $\sigma_T^\theta f \to f$ a.e. as $T \to \infty$, and the maximal operator of the θ-means $\sigma_\square^\theta$ is of weak type $(1,1)$.

Another way for proving the weak type $(1,1)$ inequality and the almost everywhere convergence is to show that $\sigma_\square^\theta$ is bounded from the Hardy space H_p to L_p, where $p<1$ and then to apply interpolation theory. Using this idea in [27] we got some new conditions on θ. More precisely, we proved that if θ is in a suitable Wiener amalgam space or in a modulation space (these spaces are often used in Gabor analysis) then we obtain almost everywhere convergence for the θ-means of $f \in L_1(\mathbb{R}^d)$. Moreover, $\sigma_\square^\theta$ is bounded from the Hardy space $H_p^\square(\mathbb{R}^d)$ to $L_p(\mathbb{R}^d)$, whenever $p_0<p<\infty$ $(p_0<1)$, and it is of weak type $(1,1)$.

In this paper we will prove sharper inequalities and convergence results. Goldberg [7] has introduced and investigated the so-called local Hardy spaces. These spaces are used also in Gabor analysis (see Gilbert and Lakey [6] and Weisz [28]). As a generalization of the results just mentioned, we will see that $\sigma_\square^\theta$ is bounded from the local Hardy space $h_p^\square(\mathbb{R}^d)$ to $L_p(\mathbb{R}^d)$ $(p_0<p<\infty)$. Note that $\|\cdot\|_{h_p^\square} \leqslant \|\cdot\|_{H_p^\square}$.

Starting with the local Hardy spaces we introduce new Wiener amalgam spaces. Under the same conditions on θ we verify that $\sigma_\square^\theta$ is bounded from the Wiener amalgam Hardy space $W(h_p^\square, \ell_\infty)$ to $W(L_p, \ell_\infty)$ $(p_0<p<\infty)$. This implies the almost everywhere convergence of the θ-means for all $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$. This is a significant generalization, because the amalgam space $W(L_1, \ell_\infty)(\mathbb{R}^d)$ is much larger than $L_1(\mathbb{R}^d)$.

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2. Wiener amalgams and Hardy spaces

Let us fix $d \geqslant 1$, $d \in \mathbb{N}$. For a set $\mathbb{Y} \neq \emptyset$ let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \cdots \times \mathbb{Y}$ taken with itself d-times. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \dots, u_d) \in \mathbb{R}^d$ set

$$u \cdot x := \sum_{k=1}^{d} u_k x_k, \qquad \|x\|_p := \left(\sum_{k=1}^{d} |x_k|^p \right)^{1/p}, \qquad |x| := \|x\|_2.$$

We briefly write $L_p^{v_s}(\mathbb{R}^d)$ instead of the weighted $L_p^{v_s}(\mathbb{R}^d, \lambda)$ space equipped with the norm (or quasi-norm)

$$\|f\|_{L_p^{v_s}} := \left(\int_{\mathbb{R}^d} |f v_s|^p \, d\lambda \right)^{1/p} \quad (0 < p \leqslant \infty),$$

where λ is the Lebesgue measure and the weight function v_s is defined by

$$v_s(\omega) := \big(1 + |\omega|\big)^s \quad \big(s \geqslant 0,\ \omega \in \mathbb{R}^d\big).$$

We use the notation $|I|$ for the Lebesgue measure of the set I.

The *weak* L_p space, $L_{p,\infty}(\mathbb{R}^d)$ $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{L_{p,\infty}} := \sup_{\rho > 0} \rho \lambda\big(|f| > \rho\big)^{1/p} < \infty,$$

while we set $L_{\infty,\infty}(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$. Note that $L_{p,\infty}(\mathbb{R}^d)$ is a quasi-normed space (see Bergh and Löfström [1]). It is easy to see that for each $0 < p \leqslant \infty$,

$$L_p\big(\mathbb{R}^d\big) \subset L_{p,\infty}\big(\mathbb{R}^d\big) \quad \text{and} \quad \|\cdot\|_{L_{p,\infty}} \leqslant \|\cdot\|_p.$$

The space of continuous functions with the supremum norm is denoted by $C(\mathbb{R}^d)$ and we will use $C_0(\mathbb{R}^d)$ for the space of continuous functions vanishing at infinity. $C_c(\mathbb{R}^d)$ denotes the space of continuous functions having compact support.

For a measurable function ϕ on $\mathbb{R}^d$ let

$$\phi_t(x) := t^{-d} \phi(x/t) \quad \big(x \in \mathbb{R}^d,\ t > 0\big).$$

Given a Schwartz function $\phi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \phi \, d\lambda \neq 0$ and $\operatorname{supp} \phi \subset [0, 1/2]^d$, the *local Hardy space* $h_p^\square(\mathbb{R}^d)$ $(0 < p \leqslant \infty)$ consists of all tempered distributions f for which

$$\|f\|_{h_p^\square(\mathbb{R}^d)} := \Big\| \sup_{0 < t < 1} |f * \phi_t| \Big\|_{L_p(\mathbb{R}^d)} < \infty.$$

If we take in the definition instead of the $L_p(\mathbb{R}^d)$ norm the $L_{p,\infty}(\mathbb{R}^d)$ norm, then we get the *weak local Hardy space* $h_{p,\infty}^\square(\mathbb{R}^d)$ $(0 < p \leqslant \infty)$. Taking the supremum over all $0 < t < \infty$ we obtain the definition of the *classical Hardy space* $H_p^\square(\mathbb{R}^d)$. Other non-zero Schwartz functions ϕ define the same spaces and equivalent norms. Usually the classical Hardy spaces are investigated. Until this time in the theory of summability methods classical Hardy spaces were considered, only (see e.g. Weisz [25] and [27] and the references therein). The local Hardy spaces were introduced in Goldberg [7]. They ensure a useful tool in Gabor analysis (see Gilbert and Lakey [6] and Weisz [28]). As we will see later, local Hardy spaces can be well used also in summation theory of Fourier transforms. Using these spaces we will get convergence results of summation methods for functions from $W(L_1, \ell_\infty)(\mathbb{R}^d)$, which is a much larger space than $L_1(\mathbb{R}^d)$. Until this time most of the convergence results were known for integrable functions, only. The classical Hardy spaces are investigated exhaustively in Stein [15] or Weisz [25] and the local Hardy spaces in Goldberg [7] and Triebel [19].

It is known that the Hardy spaces $h_p^\square(\mathbb{R}^d)$, $H_p^\square(\mathbb{R}^d)$ are equivalent to the $L_p(\mathbb{R}^d)$ space when $1 < p < \infty$. Moreover,

$$H_1^\square\big(\mathbb{R}^d\big) \subset h_1^\square\big(\mathbb{R}^d\big) \subset L_1\big(\mathbb{R}^d\big) \subset H_{1,\infty}^\square\big(\mathbb{R}^d\big) \subset h_{1,\infty}^\square\big(\mathbb{R}^d\big)$$

and

$$\|f\|_{h_{1,\infty}^\square} \leqslant \|f\|_{H_{1,\infty}^\square} \leqslant C\|f\|_1 \leqslant C\|f\|_{h_1^\square} \leqslant C\|f\|_{H_1^\square} \tag{1}$$

(see e.g. Stein [15], Weisz [25] and Goldberg [7]).

The atomic decomposition is a useful characterization of Hardy spaces. A bounded function a is an $h_p^\square(\mathbb{R}^d)$-*atom* if there exists a cube $I \subset \mathbb{R}^d$ such that

(i) $\operatorname{supp} a \subset I$,
(ii) $\|a\|_\infty \leqslant |I|^{-1/p}$,
(iii) if $|I| < 1$ then $\int_I a(x)x^k \, d\lambda(x) = 0$ for all multi-indices $k = (k_1, \dots, k_d)$ with $|k| \leqslant M$, where $M \geqslant [d(1/p - 1)]$. Note that $[x]$ denotes the integer part of $x \in \mathbb{R}$.

We will say that a is a type 1 atom if $|I| < 1$ and a type 2 atom if $|I| \geqslant 1$. If we require the moment condition in (iii) for all intervals I (or all atoms a) then we obtain the definition of $H_p^\square(\mathbb{R}^d)$*-atom* (see Lu [10], Stein [15] and Goldberg [7]).

Theorem 1. *A tempered distribution f is in $h_p^\square(\mathbb{R}^d)$ $(0 < p \leqslant 1)$ if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of $h_p^\square(\mathbb{R}^d)$-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\sum_{k=0}^\infty |\mu_k|^p < \infty \quad \text{and} \quad \sum_{k=0}^\infty \mu_k a_k = f \quad \text{in the sense of distributions.} \tag{2}$$

Moreover,

$$\|f\|_{h_p^\square} \sim \inf \left(\sum_{k=0}^\infty |\mu_k|^p \right)^{1/p}$$

where the infimum is taken over all decompositions of f of the form (2).

Given a (quasi-)Banach space X on $\mathbb{R}^d$, a measurable function f belongs to the *Wiener amalgam space* $W(X, \ell_q^{v_s})(\mathbb{R}^d)$ $(0 < q \leqslant \infty)$ if

$$\|f\|_{W(X,\ell_q^{v_s})} := \left(\sum_{k \in \mathbb{Z}^d} \|f|_{[k,k+1)}\|_X^q v_s(k)^q \right)^{1/q} < \infty$$

with the obvious modification for $q = \infty$, where $k + 1 := (k_1 + 1, \dots, k_d + 1)$. If $s = 0$ then we write simply $W(X, \ell_q)(\mathbb{R}^d)$. $W(X, c_0)(\mathbb{R}^d)$ is defined analogously, where c_0 denotes the set of bounded sequences with limit 0. In this paper we will use the Wiener amalgam spaces for $X = L_p(\mathbb{R}^d), L_{p,\infty}(\mathbb{R}^d), h_p^\square(\mathbb{R}^d), h_{p,\infty}^\square(\mathbb{R}^d)$. The closed subspace of $W(L_\infty, \ell_q)(\mathbb{R}^d)$ containing continuous functions is denoted by $W(C, \ell_q)(\mathbb{R}^d)$ $(1 \leqslant q \leqslant \infty)$. The space $W(C, \ell_1)(\mathbb{R}^d)$ is called *Wiener algebra*. It is used quite often in Gabor analysis, because it provides a convenient and general class of windows (see e.g. Walnut [22] and Gröchenig [8]). As we have seen in Feichtinger and Weisz [4,5], it plays an important role in summability theory, too. It is easy to see that

$$W(L_\infty, \ell_1)(\mathbb{R}^d) \subset L_p(\mathbb{R}^d) \subset W(L_1, \ell_\infty)(\mathbb{R}^d) \quad (1 \leqslant p \leqslant \infty)$$

and $W(L_p, \ell_p)(\mathbb{R}^d) = L_p(\mathbb{R}^d)$.

3. θ-summability of Fourier transforms

The θ-summation was considered in a great number of papers and books, such as Butzer and Nessel [3], Trigub and Belinsky [21], Bokor, Schipp, Szili and Vértesi [14,2,18,17], Natanson and Zuk [12], Weisz [23–26] and Feichtinger and Weisz [4,5].

The *Fourier transform* of $f \in L_1(\mathbb{R}^d)$ is

$$\hat{f}(x) := \int_{\mathbb{R}^d} f(t) e^{-2\pi \imath x \cdot t} \, dt \quad (x \in \mathbb{R}^d),$$

where $\imath = \sqrt{-1}$. If $f \in L_p(\mathbb{R}^d)$ for some $1 \leqslant p \leqslant 2$ then the Fourier inversion formula

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(u) e^{2\pi \imath x \cdot u} \, du \quad (x \in \mathbb{R}^d)$$

holds if $\hat{f} \in L_1(\mathbb{R}^d)$. In the theory of summation we assume that $\theta \in L_1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$. The θ-*means* of $f \in L_p(\mathbb{R}^d)$ $(1 \leqslant p \leqslant 2)$ are defined by

$$\sigma_T^\theta f(x) := \int_{\mathbb{R}^d} \theta\left(\frac{-t_1}{T_1}, \dots, \frac{-t_d}{T_d} \right) \hat{f}(t) e^{2\pi \imath x \cdot t} \, dt.$$

Then

$$\sigma_T^\theta f(x) = \int_{\mathbb{R}^d} f(x-t) K_T^\theta(t)\,dt \quad \left(x \in \mathbb{R}^d,\ T \in \mathbb{R}_+^d\right)$$

where

$$K_T^\theta(x) = \int_{\mathbb{R}^d} \theta\left(\frac{-t_1}{T_1}, \dots, \frac{-t_d}{T_d}\right) e^{2\pi \imath x \cdot t}\,dt = \left(\prod_{j=1}^d T_j\right) \hat{\theta}(T_1 x_1, \dots, T_d x_d).$$

Thus the θ-means can be rewritten as

$$\sigma_T^\theta f(x) = \left(\prod_{j=1}^d T_j\right) \int_{\mathbb{R}^d} f(x-t) \hat{\theta}(T_1 t_1, \dots, T_d t_d)\,dt. \tag{3}$$

Here we have used that $\hat{f} \in L_{p'}(\mathbb{R}^d)$ if $f \in L_p(\mathbb{R}^d)$ $(1 \leqslant p \leqslant 2, 1/p + 1/p' = 1)$ and the Hausdorff–Young inequality. Note that $\theta \in L_1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$ implies $\theta \in L_p(\mathbb{R}^d)$ $(1 \leqslant p \leqslant \infty)$.

If $\hat{\theta} \in L_1(\mathbb{R}^d)$, the definition of the *θ-means* extends to $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$ by

$$\sigma_T^\theta f := f * K_T^\theta \quad \left(T \in \mathbb{R}_+^d\right).$$

Note that $\theta \in L_1(\mathbb{R}^d)$ and $\hat{\theta} \in L_1(\mathbb{R}^d)$ imply $\theta \in C_0(\mathbb{R}^d)$.

We will investigate the *maximal operator*

$$\sigma_\square^\theta f := \sup_{T \in \mathbb{R},\, T \geqslant 1} \left|\sigma_{\mathbf{T}}^\theta f\right|,$$

where $\mathbf{T} := (T, \dots, T) \in \mathbb{R}^d$. If $\hat{\theta} \in L_1(\mathbb{R}^d)$ then (3) implies

$$\left\|\sigma_\square^\theta f\right\|_\infty \leqslant \|\hat{\theta}\|_1 \|f\|_\infty \quad \left(f \in L_\infty\left(\mathbb{R}^d\right)\right).$$

In this paper the constants C and C_p may vary from line to line and the constants C_p are depending only on p.

4. θ-summability and the $h_p^\square$ Hardy spaces

In this section we generalize the results of Section 6 in Weisz [27] from the $H_p^\square(\mathbb{R}^d)$ spaces to the $h_p^\square(\mathbb{R}^d)$ and to the Wiener amalgam Hardy spaces. More precisely, we will prove that $\sigma_\square^\theta$ is bounded from $h_p^\square(\mathbb{R}^d)$ to $L_p(\mathbb{R}^d)$ and from $W(h_p^\square, \ell_\infty)(\mathbb{R}^d)$ to $W(L_p, \ell_\infty)(\mathbb{R}^d)$. In this way we obtain almost everywhere convergence of the θ-summation method for functions from $W(L_1, \ell_\infty)(\mathbb{R}^d)$. In [27] we verified the almost everywhere convergence for $L_1(\mathbb{R}^d)$ functions, which is a much smaller space than $W(L_1, \ell_\infty)(\mathbb{R}^d)$. All results of Section 6 in [27] can be proved in this new setting, however, we present the proofs for two theorems, only, and explain the main ideas.

We denote by $B_q(c,h)$ $(c \in \mathbb{R}^d,\ h > 0)$ the ball $\{x \in \mathbb{R}^d\colon\ \|x-c\|_q < h\}$. For $B_q = B_q(x,a)$ we denote $rB_q := B_q(x, ra)$ $(r > 0)$.

We assume that the derivatives $\partial_1^{i_1} \dots \partial_d^{i_d} \hat{\theta}$ can be estimated by a non-increasing q-radial symmetric function η_m $(i_1 + \dots + i_d = m \in \mathbb{N},\ 1 \leqslant q \leqslant \infty)$, i.e.

$$\left.\begin{array}{l} \left|\partial_1^{i_1} \dots \partial_d^{i_d} \hat{\theta}\right| \leqslant \eta_m \text{ for all } i_1 + \dots + i_d = m, \\ \eta_m(x) = \eta_m(y) \text{ if } \|x\|_q = \|y\|_q, \\ \eta_m^0 \text{ is non-increasing,} \end{array}\right\} \tag{4}$$

where

$$\eta_m^0(s) := \eta_m(x) \quad \text{if } \|x\|_q = s,\ s \in \mathbb{R}^+.$$

First we show the boundedness of the maximal operator $\sigma_\square^\theta$ from $h_p^\square(\mathbb{R}^d)$ to $L_p(\mathbb{R}^d)$.

Theorem 2. *Let $0 < p \leqslant 1$, $\theta \in L_1(\mathbb{R}^d)$ and $\hat{\theta} \in L_1(\mathbb{R}^d)$ be $(N+1)$-times differentiable $(N \in \mathbb{N})$. Suppose that $\eta_m \in L_p(\mathbb{R}^d \setminus B_q(0, 1/4))$ satisfies (4) for $m = 0, N, N+1$. If $s \mapsto s^{d+m} \eta_m^0(s)$ is non-increasing for $m = 0, N$ and $s \mapsto s^{d+N+1} \eta_{N+1}^0(s)$ is non-decreasing then*

$$\left\|\sigma_\square^\theta f\right\|_p \leqslant C_p \left(\|\hat{\theta}\|_1 + \sum_{m=0,N,N+1} \|\eta_m\|_{L_p(\mathbb{R}^d \setminus B_q(0,1/4))}\right) \|f\|_{h_p^\square}$$

($f \in h_p^{\square}(\mathbb{R}^d)$). Moreover, for all $p \leqslant r < \infty$,

$$\left\|\sigma_{\square}^{\theta} f\right\|_r \leqslant C_{r,\theta} \|f\|_{h_r^{\square}} \quad \left(f \in h_r^{\square}\left(\mathbb{R}^d\right)\right). \tag{5}$$

If $p < 1$ then

$$\left\|\sigma_{\square}^{\theta} f\right\|_{L_{1,\infty}} \leqslant C_{\theta} \|f\|_1 \quad \left(f \in L_1\left(\mathbb{R}^d\right)\right). \tag{6}$$

If we suppose $\eta_m \in L_{p,\infty}(\mathbb{R}^d)$ ($m = 0, N, N+1,\ p \neq 1$) instead of $\eta_m \in L_p(\mathbb{R}^d \setminus B_q(0, 1/4))$ then

$$\left\|\sigma_{\square}^{\theta} f\right\|_{L_{p,\infty}} \leqslant C_p \|f\|_{h_p^{\square}} \quad \left(f \in h_p^{\square}\left(\mathbb{R}^d\right)\right) \tag{7}$$

and (5) and (6) hold for $p < r < \infty$.

Proof. Let a be an arbitrary $h_p(\mathbb{R}^d)$-atom with support I, where I is a cube and $2^{-K-1} < |I|^{1/d} \leqslant 2^{-K}$ for some $K \in \mathbb{Z}$. We may suppose that the center of I is zero. If a is a type 1 atom, i.e. $K \geqslant 0$, then

$$\left\|\sigma_{\square}^{\theta} a\right\|_p \leqslant C_p\left(\|\hat{\theta}\|_1 + \|\eta_N\|_{L_p(\mathbb{R}^d \setminus B_q(0,1/4))} + \|\eta_{N+1}\|_{L_p(\mathbb{R}^d \setminus B_q(0,1/4))}\right)$$

and the corresponding inequality to (7) were proved in [27]. Now suppose that a is a type 2 atom with $K < 0$. Let B_q be the smallest ball which contains I. Then

$$\left|\sigma_{\mathbf{T}}^{\theta} a(x)\right| = T^d \left|\int_{B_q} a(t) \hat{\theta}\left(T(x-t)\right) dt\right| \leqslant C_p T^d 2^{Kd/p} \int_{B_q} \eta_0\left(T(x-t)\right) dt.$$

Since the function $s \mapsto s^d \eta_0^0(s)$ is non-increasing, we get

$$\begin{aligned}
\sup_{T \geqslant 1}\left|\sigma_{\mathbf{T}}^{\theta} a(x)\right| &\leqslant C_p 2^{Kd/p} \int_{B_q} \eta_0(x-t)\, dt \\
&\leqslant C_p 2^{Kd/p} \int_{B_q} \eta_0^0\left(i 2^{-K}\right) dt \leqslant C_p 2^{Kd/p - Kd} \eta_0^0\left(i 2^{-K}\right)
\end{aligned}$$

where $x \in B_q(0, (i+2)2^{-K}) \setminus B_q(0, (i+1)2^{-K})$. Hence

$$\begin{aligned}
\int_{\mathbb{R}^d \setminus (16 B_q)} \left|\sigma_{\square}^{\theta} a(x)\right|^p dx &\leqslant \sum_{i=4}^{\infty} \int_{B_q(0,(i+2)2^{-K}) \setminus B_q(0,(i+1)2^{-K})} \left|\sigma_{\square}^{\theta} a(x)\right|^p dx \\
&\leqslant C_p \sum_{i=4}^{\infty} 2^{-Kd} i^{d-1} 2^{Kd - Kdp} \eta_0^0\left(i 2^{-K}\right)^p \\
&\leqslant C_p 2^{Kd(1-p)} \int_{\mathbb{R}^d \setminus B_q(0,1)} \eta_0(t)^p\, dt \\
&\leqslant C_p \int_{\mathbb{R}^d \setminus B_q(0,1)} \eta_0(t)^p\, dt,
\end{aligned}$$

because $K < 0$.

Let us introduce the set

$$E_\rho := \left\{i \geqslant 4:\ \eta_0^0\left(i 2^{-K}\right) > \rho C_p^{-1} 2^{-Kd/p + Kd}\right\}$$

for a moment. To prove (7) observe that

$$\rho^p \lambda\left(\left\{\sigma_{\square}^{\theta} a > \rho\right\} \cap \left\{\mathbb{R}^d \setminus (16 B_q)\right\}\right) \leqslant C_p \rho^p \sum_{i \in E_\rho} i^{d-1} 2^{-Kd}.$$

If k is the largest integer, for which $\eta_0^0(k2^{-K}) > \rho C_p^{-1} 2^{-Kd/p+Kd}$, then

$$\begin{aligned}\rho^p \lambda\left(\left\{\sigma_\square^\theta a > \rho\right\} \cap \left\{\mathbb{R}^d \setminus (16B_q)\right\}\right) &\leqslant C_p \rho^p 2^{-Kd} k^d \\ &\leqslant C_p \rho^p \lambda\left(\left\{\eta_0 > \rho C_p^{-1} 2^{-Kd/p+Kd}\right\}\right) \\ &\leqslant C_p 2^{Kd(1-p)} \|\eta_0\|_{L_{p,\infty}}.\end{aligned}$$

The theorem follows from the atomic decomposition in the usual way. $\square$

Combining the idea just presented and the proof of Theorem 6.2 in [27], we can prove the next result.

Theorem 3. *Suppose that $\theta \in L_1^{v_1}(\mathbb{R}^d)$ and $\eta_0 \in L_1(\mathbb{R}^d \setminus B_q(0,1/4))$ satisfies (4). If the function $s \mapsto s^d \eta_0^0(s)$ is non-increasing then*

$$\left\|\sigma_\square^\theta f\right\|_1 \leqslant C\|f\|_{h_1^\square} \quad \left(f \in h_1^\square\left(\mathbb{R}^d\right)\right). \tag{8}$$

As in [27], if $\theta \in L_1^{v_1}(\mathbb{R}) \cap V_1^2(\mathbb{R})$ or $\theta \in L_1^{v_1}(\mathbb{R}^d) \cap M_1^{v_s}(\mathbb{R}^d)$ for some $s > d$, then (8) holds. For the definitions of the *Sobolev-type space* $V_1^2(\mathbb{R})$ and *weighted modulation space* $M_1^{v_s}(\mathbb{R}^d)$ see [27].

Now we are ready to prove our main result. Note that we will use the exact values of the constants in (9) and (11).

Theorem 4. *Let $0 < p \leqslant 1$, $\theta \in L_1(\mathbb{R}^d)$ and $\hat{\theta} \in L_1(\mathbb{R}^d)$ be $(N+1)$-times differentiable ($N \in \mathbb{N}$). Suppose that $\eta_m \in L_p(\mathbb{R}^d \setminus B_q(0,1/4))$ satisfies (4) for $m = 0, N, N+1$. If $s \mapsto s^{d+m}\eta_m^0(s)$ is non-increasing for $m = 0, N$ and $s \mapsto s^{d+N+1}\eta_{N+1}^0(s)$ is non-decreasing then*

$$\left\|\sigma_\square^\theta f\right\|_{W(L_p,\ell_\infty)} \leqslant C_p\left(\|\hat{\theta}\|_1 + \sum_{m=0,N,N+1} \|\eta_m\|_{L_p(\mathbb{R}^d\setminus B_q(0,1/4))}\right)\|f\|_{W(h_p^\square,\ell_\infty)} \tag{9}$$

($f \in W(h_p^\square,\ell_\infty)(\mathbb{R}^d)$). Moreover, for all $p \leqslant r < \infty$,

$$\left\|\sigma_\square^\theta f\right\|_{W(L_r,\ell_\infty)} \leqslant C_{r,\theta}\|f\|_{W(h_r^\square,\ell_\infty)} \quad \left(f \in W\left(h_r^\square,\ell_\infty\right)\left(\mathbb{R}^d\right)\right). \tag{10}$$

If $p < 1$ then

$$\left\|\sigma_\square^\theta f\right\|_{W(L_{1,\infty},\ell_\infty)} \leqslant C_p\left(\|\hat{\theta}\|_1 + \sum_{m=0,N,N+1} \|\eta_m\|_{L_p(\mathbb{R}^d\setminus B_q(0,1/4))}\right)^p \|\hat{\theta}\|_1^{1-p}\|f\|_{W(L_1,\ell_\infty)} \tag{11}$$

for all $f \in W(L_1,\ell_\infty)(\mathbb{R}^d)$.

Proof. Obviously,

$$\int_k^{k+1} \left|\sigma_\square^\theta f(x)\right|^p dx \leqslant \sum_{j\in\mathbb{Z}^d} \int_k^{k+1} \left|\sigma_\square^\theta (f|_{[j,j+1)})(x)\right|^p dx$$

for each $k \in \mathbb{Z}^d$. Let $f|_{[j,j+1)} = \sum_{l=0}^\infty \mu_{j,l} a_{j,l}$ be an atomic decomposition of $f|_{[j,j+1)} \in h_p^\square(\mathbb{R}^d)$ such that

$$\sum_{l=0}^\infty |\mu_{j,l}|^p \leqslant C_p \|f|_{[j,j+1)}\|_{h_p^\square}^p.$$

Since $\operatorname{supp}\phi \subset [0,1/2]^d$ we have $\operatorname{supp} f|_{[j,j+1)} * \phi_t \subset [j, j+3/2)$, where $j \in \mathbb{Z}^d$, $0 < t < 1$. So we may suppose that $\operatorname{supp} a_{j,l} \subset [j, j+3/2)$. Thus

$$\int_k^{k+1} \left|\sigma_\square^\theta f(x)\right|^p dx \leqslant \sum_{j\in\mathbb{Z}^d} \sum_{l=0}^\infty |\mu_{j,l}|^p \int_k^{k+1} \left|\sigma_\square^\theta a_{j,l}(x)\right|^p dx. \tag{12}$$

For simplicity we assume that each coordinate of $j-k$ is non-negative, i.e. $j_l \geqslant k_l$ $(l = 0,\dots,d)$. Denote one of the atoms $a_{j,l}$ by a and suppose that it is supported in a cube I with $2^{-K-1} < |I|^{1/d} \leqslant 2^{-K}$ for some $K \in \mathbb{Z}$. Then $I \subset [j, j+3/2)$. Let

B_q be the smallest ball which contains I. If $\|j-k\|_q \leqslant 2d^{1/q}$ then

$$\int_{(k,k+1)\cap(4d^{1/q})B_q} \left|\sigma_\square^\theta a(x)\right|^p dx \leqslant \int_{(4d^{1/q})B_q} \sup_{T\geqslant 1}\left|\int_{\mathbb{R}^d} a(t)T^d\hat\theta\big(T(x-t)\big)\,dt\right|^p dx$$
$$\leqslant C_p 2^{Kd}2^{-Kd}\|\hat\theta\|_1^p.$$

On the other hand,

$$\int_{(k,k+1)\cap((4d^{1/q})B_q)^c} \left|\sigma_\square^\theta a(x)\right|^p dx \leqslant \int_{((4d^{1/q})B_q)^c} \left|\sigma_\square^\theta a(x)\right|^p dx$$
$$\leqslant C_p\Bigg(\sum_{m=0,N,N+1}\|\eta_m\|_{L_p(\mathbb{R}^d\setminus B_q(0,1/4))}\Bigg)^p$$

was proved in [27] if a is a type 1 atom. It is easy to see that the last expression is 0 if a is a type 2 atom, because $K<0$. Then

$$\sum_{j:\ \|j-k\|_q\leqslant 2d^{1/q}}\sum_{l=0}^\infty |\mu_{j,l}|^p\int_k^{k+1}\left|\sigma_\square^\theta a_{j,l}(x)\right|^p dx \leqslant C_p'\sum_{j:\ \|j-k\|_q\leqslant 2d^{1/q}}\sum_{l=0}^\infty|\mu_{j,l}|^p$$
$$\leqslant C_p'\sum_{j:\ \|j-k\|_q\leqslant 2d^{1/q}}\|f|_{[j,j+1)}\|_{h_p^\square}^p$$
$$\leqslant C_p'\sup_{j\in\mathbb{Z}^d}\|f|_{[j,j+1)}\|_{h_p^\square}^p,$$

where $C_p' := C_p(\|\hat\theta\|_1+\sum_{m=0,N,N+1}\|\eta_m\|_{L_p(\mathbb{R}^d\setminus B_q(0,1/4))})^p$.

If $\|j-k\|_q > 2d^{1/q}$ then for a type 1 atom

$$\int_k^{k+1}\left|\sigma_\square^\theta a(x)\right|^p dx \leqslant \int_k^{k+1}\sup_{T\geqslant 2^K}\left|\sigma_{\mathbf{T}}^\theta a(x)\right|^p dx + \int_k^{k+1}\sup_{1\leqslant T<2^K}\left|\sigma_{\mathbf{T}}^\theta a(x)\right|^p dx$$
$$=: (A)+(B).$$

By the definition of the atom and by Taylor's formulae

$$\sigma_{\mathbf{T}}^\theta a(x) = T^d\int_I a(t)\hat\theta\big(T(x-t)\big)\,dt$$
$$= T^d\sum_{i_1+\cdots+i_d=N}(-1)^N\int_I a(t)\big(\partial_1^{i_1}\ldots\partial_d^{i_d}\hat\theta\big(T(x-c)-Tv(t-c)\big)\big)T^N\prod_{j=1}^d\frac{(t_j-c_j)^{i_j}}{i_j!}\,dt,$$

where $0<v<1$ and we may suppose that $c=j+3/4$ is the center of B_q. Then

$$\left|\sigma_{\mathbf{T}}^\theta a(x)\right| \leqslant C_pT^{d+N}2^{Kd/p}2^{-KN}\int_I\eta_N\big[T\big(x-c-v(t-c)\big)\big]\,dt.$$

Since the function $s\mapsto s^{N+d}\eta_N^0(s)$ is non-increasing, we get

$$\sup_{T\geqslant 2^K}\left|\sigma_{\mathbf{T}}^\theta a(x)\right| \leqslant C_p2^{K(d+N)}2^{Kd/p}2^{-KN}\int_I\eta_N\big[2^K\big(x-c-v(t-c)\big)\big]\,dt.$$

Thus

$$(A) \leqslant C_p2^{Kdp+Kd}\int_k^{k+1}\left|\int_I\eta_N\big[2^K\big(x-c-v(t-c)\big)\big]\,dt\right|^p dx$$
$$= C_p2^{Kdp+Kd}\int_0^1\left|\int_I\eta_N\big[2^K\big(x+k-c-v(t-c)\big)\big]\,dt\right|^p dx.$$

Observe that

$$\left\|c+v(t-c)-k-x\right\|_q \geqslant \left\|c+v(t-c)-k\right\|_q-\|x\|_q \geqslant \|j-k\|_q-d^{1/q} \geqslant \frac{\|j-k\|_q}{2}.$$

Since η_N^0 is non-increasing, we conclude

$$(A) \leqslant C_p 2^{Kdp+Kd} \int_0^1 \left|\int_I \eta_N\left(2^{K-1}\|j-k\|_q\right) dt\right|^p dx \leqslant C_p 2^{Kd}\left|\eta_N^0\left(2^{K-1}\|j-k\|_q\right)\right|^p.$$

The inequality

$$(B) \leqslant C_p 2^{Kd}\left|\eta_{N+1}^0\left(2^{K-1}\|j-k\|_q\right)\right|^p$$

can be proved in the same way. If a is a type 2 atom, then we have to investigate (A), only, and we get without any differentiation that

$$(A) \leqslant C_p 2^{Kd}\left|\eta_0^0\left(2^{K-1}\|j-k\|_q\right)\right|^p.$$

This implies that

$$\begin{aligned}
&\sum_{j:\ \|j-k\|_q>2d^{1/q}} \sum_{l=0}^{\infty} |\mu_{j,l}|^p \int_k^{k+1} \left|\sigma_\square^\theta a_{j,l}(x)\right|^p dx \\
&\quad\leqslant C_p \sum_{j:\ \|j-k\|_q>2d^{1/q}} \sum_{l=0}^{\infty} |\mu_{j,l}|^p 2^{Kd} \sum_{m=0,N,N+1} \left|\eta_m^0\left(2^{K-1}\|j-k\|_q\right)\right|^p \\
&\quad\leqslant C_p \sum_{j:\ \|j-k\|_q>2d^{1/q}} \|f|_{[j,j+1)}\|_{h_p^\square}^p 2^{Kd} \sum_{m=0,N,N+1} \left|\eta_m^0\left(2^{K-1}\|j-k\|_q\right)\right|^p \\
&\quad\leqslant C_p \|f\|_{W(h_p^\square,\ell_\infty)}^p \sum_{n\neq 0,\, n_l\geqslant 0} 2^{Kd} \sum_{m=0,N,N+1} \left|\eta_m^0\left(2^{K-1}\|n\|_q\right)\right|^p.
\end{aligned}$$

Let

$$c_n := (c_{n,l};\ l=1,\ldots,d) \quad \text{and} \quad c_{n,l} := \begin{cases} n_l-1/2, & n_l>0, \\ 0, & n_l=0. \end{cases}$$

Observe that the balls $B_q(c_n,1/4)$ are pairwise disjoint. Moreover, if $t \in B_q(c_n,1/4)$ then $\|t\|_q \leqslant \|n\|_q$ and so $\eta_m^0(2^{K-1}\|n\|_q) \leqslant \eta_m(2^{K-1}t)$. Thus

$$\begin{aligned}
\sum_{n\neq 0,\, n_l\geqslant 0} 2^{Kd}\left|\eta_m^0\left(2^{K-1}\|n\|_q\right)\right|^p &\leqslant C 2^{Kd} \sum_{n\neq 0,\, n_l\geqslant 0} \int_{B_q(c_n,1/4)} \eta_m\left(2^{K-1}t\right)^p dt \\
&\leqslant C 2^{Kd} \int_{\mathbb{R}^d\setminus B_q(0,1/4)} \eta_m\left(2^{K-1}t\right)^p dt \\
&= C \int_{\mathbb{R}^d\setminus B_q(0,1/4)} \eta_m(t)^p dt. \qquad (13)
\end{aligned}$$

Taking into account (12) and (13) we finished the proof of (9).

Since $h_\infty^\square(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$ and $W(L_\infty,\ell_\infty)(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$, (9) holds for $p=\infty$, too. In other words,

$$\sigma_\square^\theta : W\left(h_p^\square,\ell_\infty\right)\left(\mathbb{R}^d\right) \longrightarrow W(L_p,\ell_\infty)\left(\mathbb{R}^d\right) \qquad (14)$$

and

$$\sigma_\square^\theta : W\left(h_\infty^\square,\ell_\infty\right)\left(\mathbb{R}^d\right) \longrightarrow W(L_\infty,\ell_\infty)\left(\mathbb{R}^d\right) \qquad (15)$$

are bounded operators. By real interpolation we obtain that

$$\sigma_\square^\theta : W\left(\left(h_p^\square,h_\infty^\square\right)_{\eta,\infty},\ell_\infty\right)\left(\mathbb{R}^d\right) \longrightarrow W\left((L_p,L_\infty)_{\eta,\infty},\ell_\infty\right)\left(\mathbb{R}^d\right)$$

is bounded, where $0 < \eta < 1$ is arbitrary. Here we have used that

$$\big(W\big(h_p^\square, \ell_\infty\big)\big(\mathbb{R}^d\big), W\big(h_\infty^\square, \ell_\infty\big)\big(\mathbb{R}^d\big)\big)_{\eta,\infty} = W\big(\big(h_p^\square, h_\infty^\square\big)_{\eta,\infty}, \ell_\infty\big)\big(\mathbb{R}^d\big)$$

and the analogue for L_p (see Sagher [13], Kisliakov and Xu [9], Berg and Löfström [1]). If $p < 1$ then the choice $\eta = 1 - p$ implies the boundedness of

$$\sigma_\square^\theta : W\big(h_{1,\infty}^\square, \ell_\infty\big)\big(\mathbb{R}^d\big) \longrightarrow W(L_{1,\infty}, \ell_\infty)\big(\mathbb{R}^d\big)$$

and inequality (1) proves (11). Since η_m^0 is non-increasing, $\eta_m \in L_p(\mathbb{R}^d \setminus B_q(0, 1/4))$ implies easily that $\eta_m \in L_r(\mathbb{R}^d \setminus B_q(0, 1/4))$ for all $p \leqslant r$, which shows (10) for $p \leqslant r \leqslant 1$.

Using complex interpolation we conclude similarly to the method of Triebel [20] that

$$\big(W\big(h_1^\square, \ell_\infty\big)\big(\mathbb{R}^d\big), W\big(h_\infty^\square, \ell_\infty\big)\big(\mathbb{R}^d\big)\big)_{[\eta]} = W\big(\big(h_1^\square, h_\infty^\square\big)_{[\eta]}, \ell_\infty\big)\big(\mathbb{R}^d\big)$$

and that the same holds with L_1 instead of $h_1^\square$. Applying (14) for $p = 1$ and (15) we get that for arbitrary $0 < \eta < 1$

$$\sigma_\square^\theta : W\big(\big(h_1^\square, h_\infty^\square\big)_{[\eta]}, \ell_\infty\big)\big(\mathbb{R}^d\big) \longrightarrow W\big((L_1, L_\infty)_{[\eta]}, \ell_\infty\big)\big(\mathbb{R}^d\big)$$

is bounded. Choosing $\eta = 1 - 1/r$ $(1 < r < \infty)$ we obtain the boundedness of

$$\sigma_\square^\theta : W\big(h_r^\square, \ell_\infty\big)\big(\mathbb{R}^d\big) \longrightarrow W(L_r, \ell_\infty)\big(\mathbb{R}^d\big),$$

in other words (10). This finishes the proof of Theorem 4. □

The next result follows easily from Theorems 2 and 4.

Corollary 1. *Let $\theta \in L_1(\mathbb{R}^d)$ and $\hat\theta \in L_1(\mathbb{R}^d)$ be $(N+1)$-times differentiable $(N \in \mathbb{N})$. Assume that there exists $d + N < \beta \leqslant d + N + 1$ such that*

$$\big|\partial_1^{i_1} \ldots \partial_d^{i_d} \hat\theta(x)\big| \leqslant C\|x\|_q^{-\beta} \quad (x \neq 0)$$

whenever $i_1 + \cdots + i_d = 0$, N or $N + 1$. Then (5), (6), (10) *and* (11) *hold for all $d/\beta < r < \infty$ and* (7) *for $p = d/\beta$.*

As we have seen in [27] if $\theta \in L_1^{v_1}(\mathbb{R}) \cap V_1^2(\mathbb{R})$ and $\theta'' \in L_1^{v_1}(\mathbb{R})$ then the conditions of Corollary 1 are fulfilled with $\beta = 2$, $d = 1$ and $N = 0$. Observe that if θ has compact support then the conditions $\theta \in L_1^{v_1}(\mathbb{R})$ and $\theta'' \in L_1^{v_1}(\mathbb{R})$ follow automatically.

Since $W(L_1, \ell_\infty)(\mathbb{R}^d) \supset L_1(\mathbb{R}^d)$, the next corollary is much more general than the results in [27].

Corollary 2. *If $p < 1$ in Theorem 4 or Corollary 1 then*

$$\lim_{T\to\infty} \sigma_{\mathbf{T}}^\theta f = \theta(0) f \quad \text{a.e.}$$

for all $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$.

Proof. For $f \in C_c(\mathbb{R}^d)$ we obtain the convergence from

$$\left|\sigma_{\mathbf{T}}^\theta f(x) - \left(\int_{\mathbb{R}^d} \hat\theta \, d\lambda\right) f(x)\right| \leqslant \int_{\mathbb{R}^d} \left| f\left(x - \frac{t}{T}\right) - f(x)\right| \big|\hat\theta(t)\big| \, dt$$

and from Lebesgue dominated convergence theorem. Since $C_c(\mathbb{R}^d)$ is dense in $W(L_1, c_0)(\mathbb{R}^d)$, the corollary follows for all $f \in W(L_1, c_0)(\mathbb{R}^d)$ from (11) and from the usual density argument due to Marcinkiewicz and Zygmund [11]. If $\hat\theta$ has compact support $[-c, c]$ and $x \in [-k, k]$ for some $k \in \mathbb{N}^d$ than it is easy to see that

$$\sigma_{\mathbf{T}}^\theta f(x) = \sigma_{\mathbf{T}}^\theta (f|_{[-c-k.c+k]})(x) \quad (T \geqslant 1).$$

However, for $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$ we have $f|_{[-c-k.c+k]} \in W(L_1, c_0)(\mathbb{R}^d)$ and so the corollary holds for all $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$, whenever $\hat\theta$ has compact support.

Now suppose that the support of $\hat\theta$ is not compact. Observe that Theorem 4 can be applied for the functions $\hat\theta|_{B_q(0,k)}$ $(k \in \mathbb{N})$ instead of $\hat\theta$ with the same functions η_m. Hence, using the notation

$$\sigma_{\mathbf{T},\hat\theta} f(x) = \int_{\mathbb{R}^d} f\left(x - \frac{t}{T}\right) \hat\theta(t) \, dt$$

we obtain

$$\lim_{T\to\infty} \sigma_{\mathbf{T},\hat{\theta}|_{B_q(0,k)}} f = \Big(\int_{\mathbb{R}^d} \hat{\theta}|_{B_q(0,k)}\, d\lambda\Big) f \quad \text{a.e.}$$

for all $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$.

Fix $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$ and set

$$\xi := \limsup_{T\to\infty} \big|\sigma_{\mathbf{T}}^\theta f - \theta(0) f\big|.$$

It is sufficient to show that $\xi = 0$ a.e. We have

$$\begin{aligned}
\xi &\leqslant \limsup_{T\to\infty} |\sigma_{\mathbf{T},\hat{\theta}} f - \sigma_{\mathbf{T},\hat{\theta}|_{B_q(0,k)}} f| + \limsup_{T\to\infty} \Big|\sigma_{\mathbf{T},\hat{\theta}|_{B_q(0,k)}} f - \Big(\int_{\mathbb{R}^d} \hat{\theta}|_{B_q(0,k)}\, d\lambda\Big) f\Big| \\
&\quad + \Big|\Big(\int_{\mathbb{R}^d} \hat{\theta}|_{B_q(0,k)}\, d\lambda\Big) f - \Big(\int_{\mathbb{R}^d} \hat{\theta}\, d\lambda\Big) f\Big| \\
&\leqslant \limsup_{T\to\infty} |\sigma_{\mathbf{T},\hat{\theta}-\hat{\theta}|_{B_q(0,k)}} f| + \Big(\int_{\mathbb{R}^d\setminus B_q(0,k)} |\hat{\theta}|\, d\lambda\Big) |f|
\end{aligned}$$

for all $k \in \mathbb{N}$. By Theorem 4 we conclude

$$\begin{aligned}
\|\xi\|_{W(L_{1,\infty},\ell_\infty)} &\leqslant \Big\| \sup_{T\geqslant 1} |\sigma_{\mathbf{T},\hat{\theta}-\hat{\theta}|_{B_q(0,k)}} f| \Big\|_{W(L_{1,\infty},\ell_\infty)} + \Big(\int_{\mathbb{R}^d\setminus B_q(0,k)} |\hat{\theta}|\, d\lambda\Big) \|f\|_{W(L_{1,\infty},\ell_\infty)} \\
&\leqslant C_p \Big(\|\hat{\theta} - \hat{\theta}|_{B_q(0,k)}\|_1 + \sum_{m=0,N,N+1} \|\eta_m\|_{L_p(\mathbb{R}^d\setminus B_q(0,1/4))}\Big)^p \|\hat{\theta} - \hat{\theta}|_{B_q(0,k)}\|_1^{1-p} \|f\|_{W(L_1,\ell_\infty)} \\
&\quad + \|\hat{\theta} - \hat{\theta}|_{B_q(0,k)}\|_1 \|f\|_{W(L_1,\ell_\infty)}
\end{aligned}$$

for all $k \in \mathbb{N}$. Since $\|\hat{\theta} - \hat{\theta}|_{B_q(0,k)}\|_1 \to 0$ as $k \to \infty$, $\|\xi\|_{W(L_{1,\infty},\ell_\infty)} = 0$ and so $\xi = 0$ a.e., which finishes the proof. □

In [27] we have listed 20 examples as special cases of θ-summations, such as Weierstrass, Picard, Bessel, de La Vallée-Poussin, Rogosinski, Riesz and Riemann summations.

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